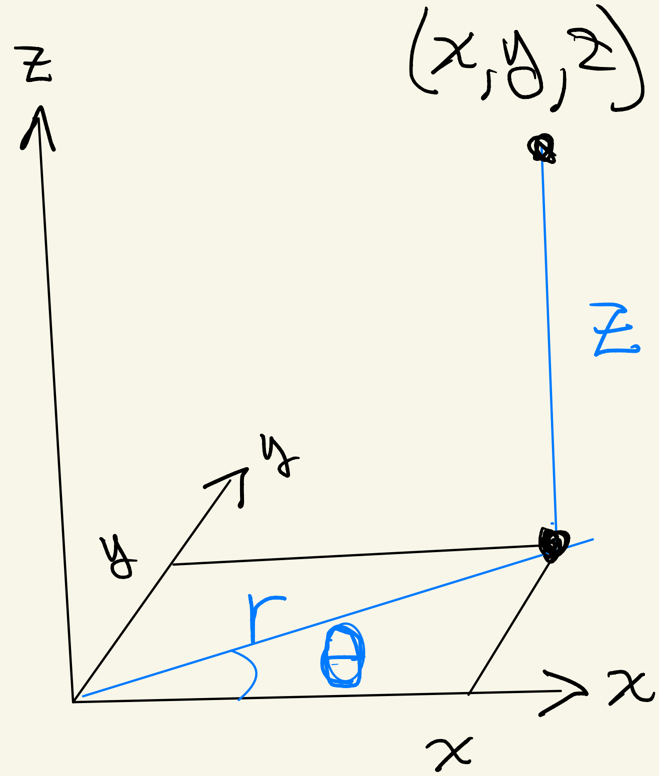
§ 15.6 Spherical / Cylindrical Coords

Cylindrical coords are easier but spherical coords are more important

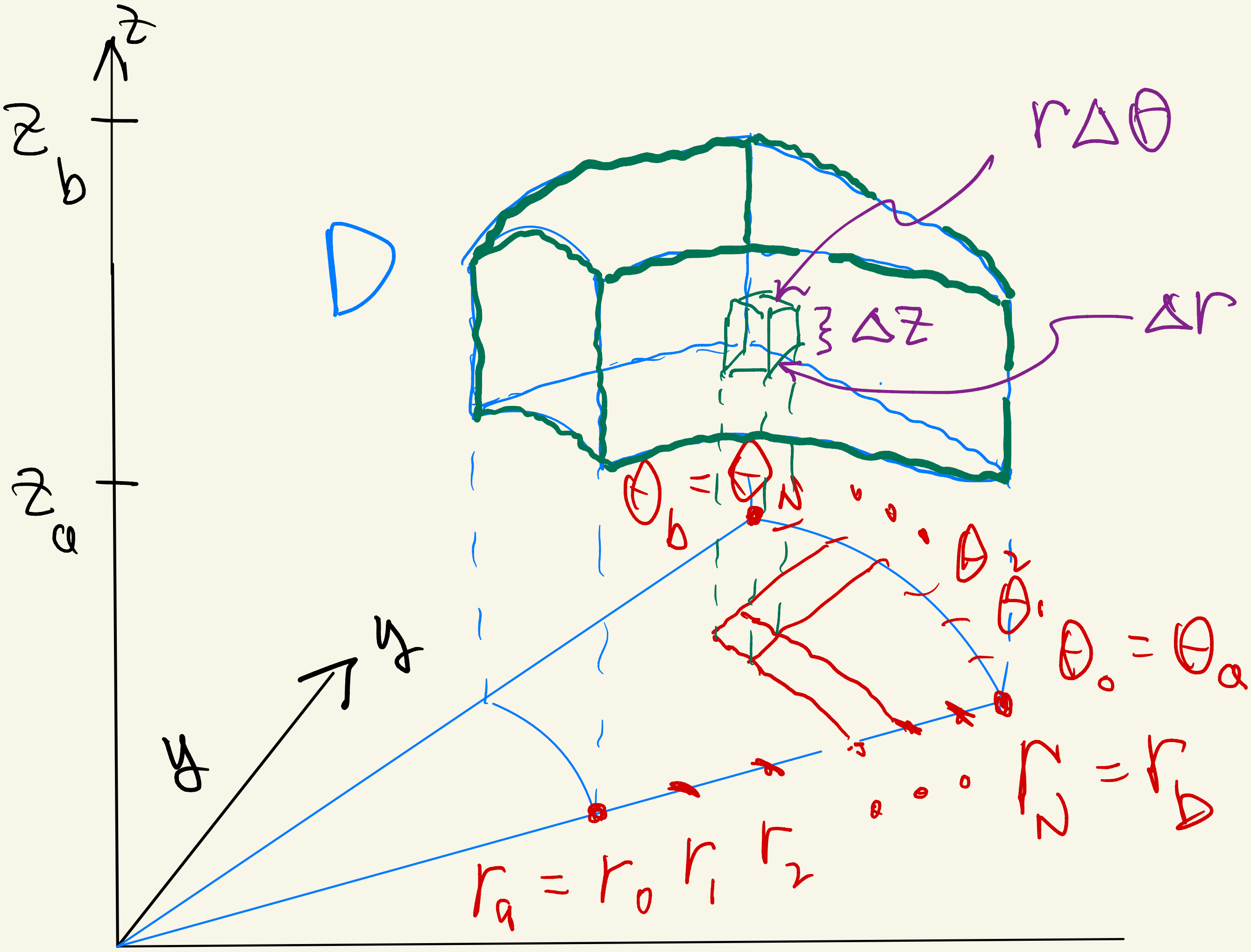
- Cylindrical coords —

"Uses polar coords for x and y, but keep z unchanged"



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

Idea: To Change variables of Int. write Riemann Sum in terms of (r, θ, z)



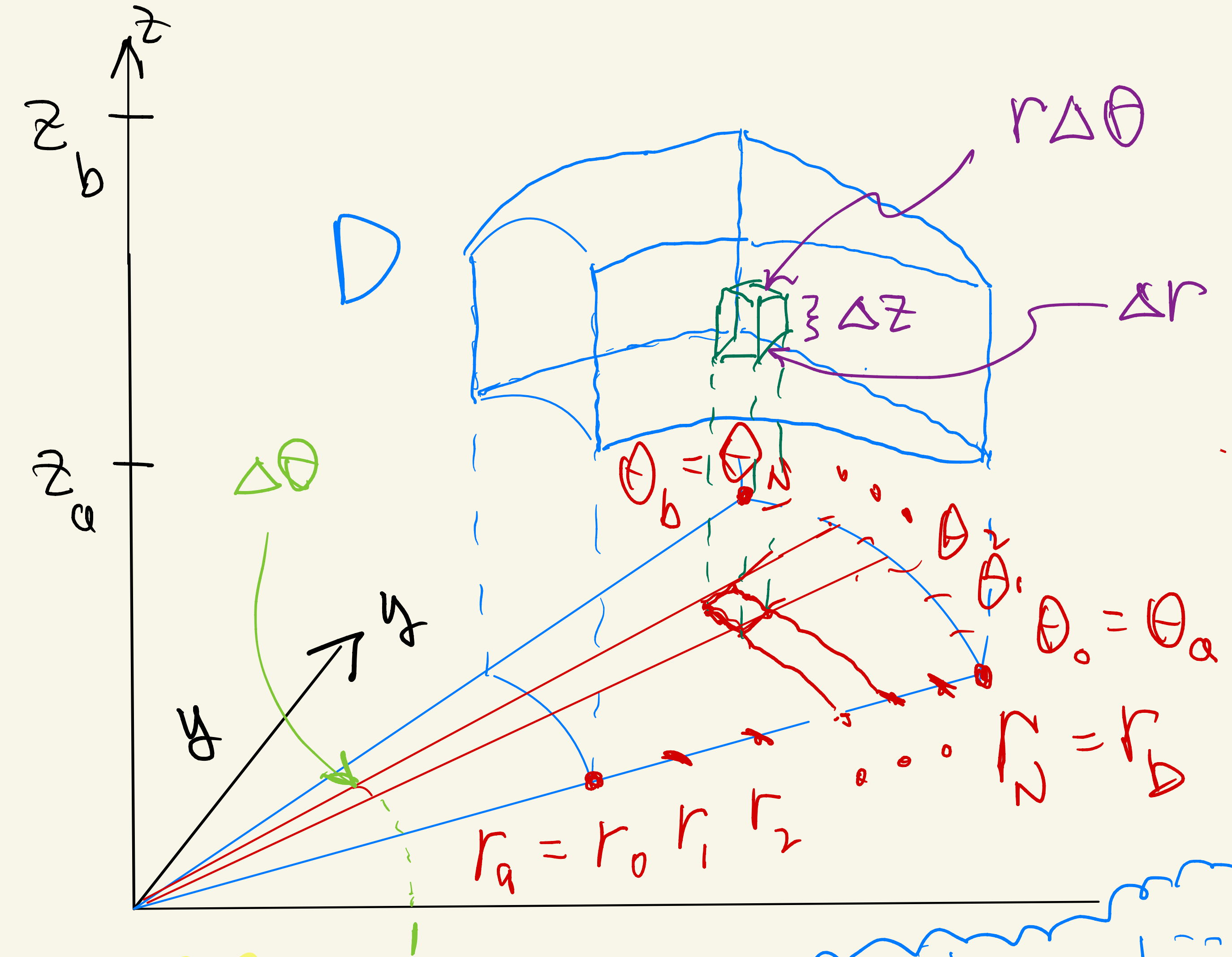
- Start with D in (x, y, z) coords
- Discretize in (r, θ, z)

$$\theta_a = \theta_0 < \theta_1 < \cdots < \theta_N = \theta_b \qquad \Delta\theta = \frac{\theta_b - \theta_a}{N}$$

$$r_a = r_0 < r_1 < \cdots < r_N = r_b \qquad \Delta r = \frac{r_b - r_a}{N}$$

$$z_a = z_0 < z_1 < \cdots < z_N = z_b \qquad \Delta z = \frac{z_b - z_a}{N}$$

Express Riemann Sum in (r, θ, z)



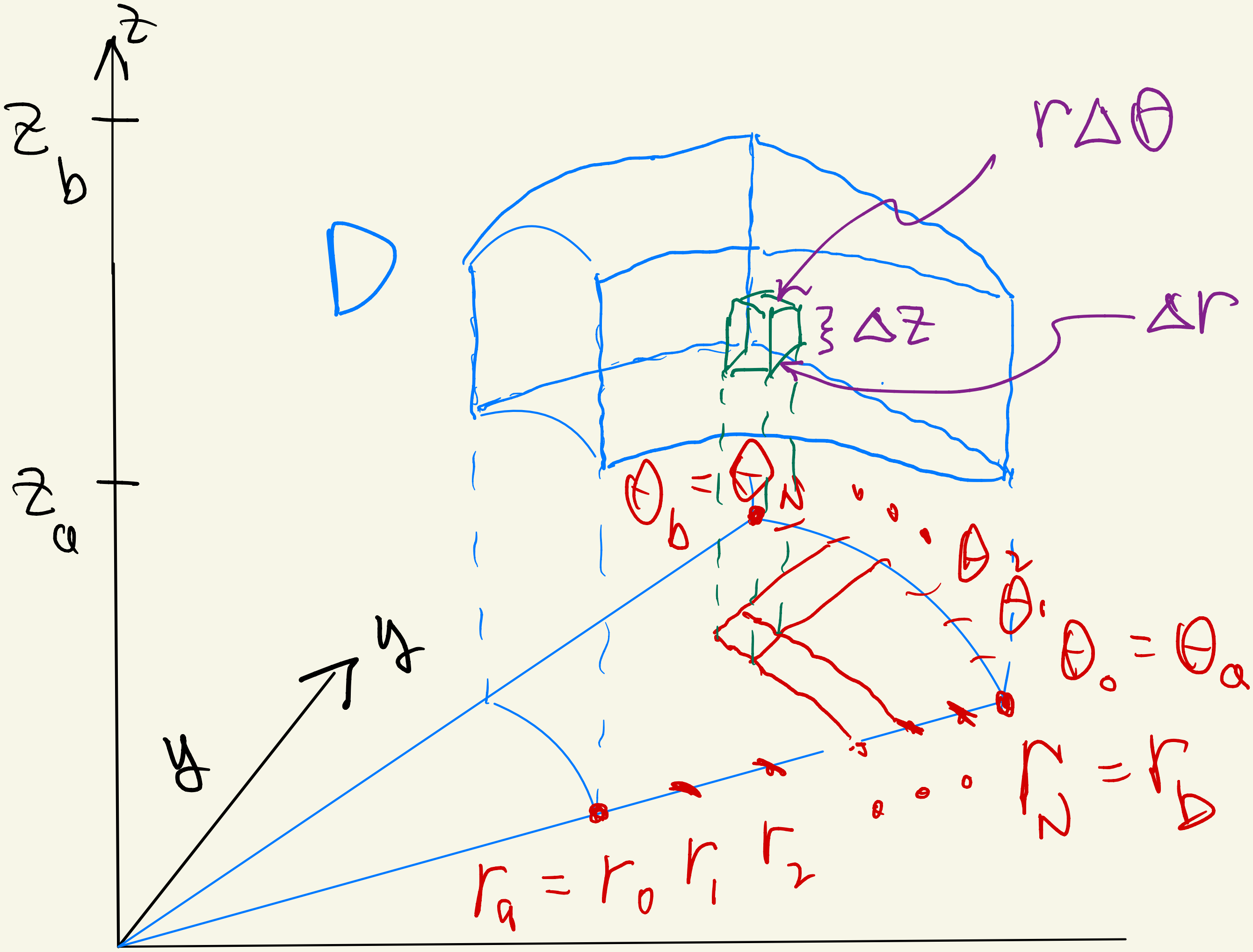
$$I = \iiint_D f(x,y,z)\,dV$$

Write as limit of Riemann Sum

$$= \lim_{N\to\infty} \sum_{(x_i,y_j,z_k)\in D} f(x_i,y_j,z_k)\,\Delta x\Delta y\Delta z$$

$$= \lim_{N\to\infty} \sum_{(r_i,\theta_j,z_k)\in D_{r\theta z}} f(r_i\cos\theta_j, r_i\sin\theta_j, z_k)\,A\,\Delta r\Delta\theta\Delta z$$

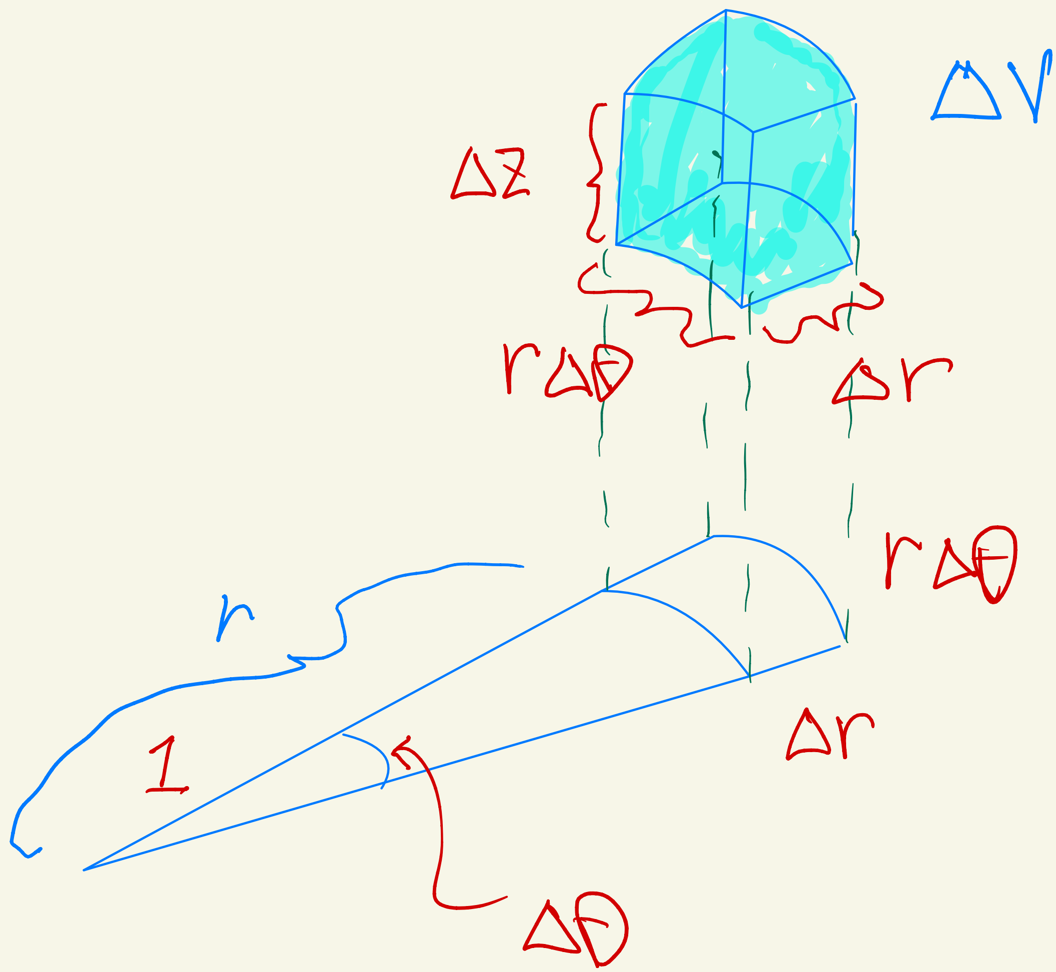
Amplification factor for Vol



$$\Delta x \Delta y \Delta z = \Delta V = A \, \Delta r \Delta \theta \Delta z$$

Amplification factor for volume

To get the amplification factor for Volume, blow up the rectangle $\Delta r \Delta \theta \Delta z$ . . .



Conclude from the geometry:

$$\Delta x \Delta y \Delta z = \Delta V \approx r \, \Delta r \Delta \theta \Delta z$$

Amplification factor $\boxed{A = r}$

We say: $dx\,dy\,dz = dV = r\,dr\,d\theta\,dz$

Conclude:

$$\iiint_{D_{xyz}} f(x,y,z)\,dV$$

$$= \lim_{N\to\infty} \sum_{(x_i, y_j, z_k)\in D_{xyz}} f(x_i, y_j, z_k)\,\Delta x\,\Delta y\,\Delta z$$

$$= \lim_{N\to\infty} \sum_{(r_i, \theta_j, z_k)\in D_{r\theta z}} f(r_i\cos\theta_j, r_i\sin\theta_j, z_k)\,\underset{\sim}{r_k}\,\Delta r\,\Delta\theta\,\Delta z$$

A

(Riemann Sum in (r, θ, z))

$$= \iiint_{D_{r\theta z}} f(r\cos\theta, r\sin\theta, z)\,\underbrace{r\,dr\,d\theta\,dz}_{dA = dx\,dy\,dz}$$

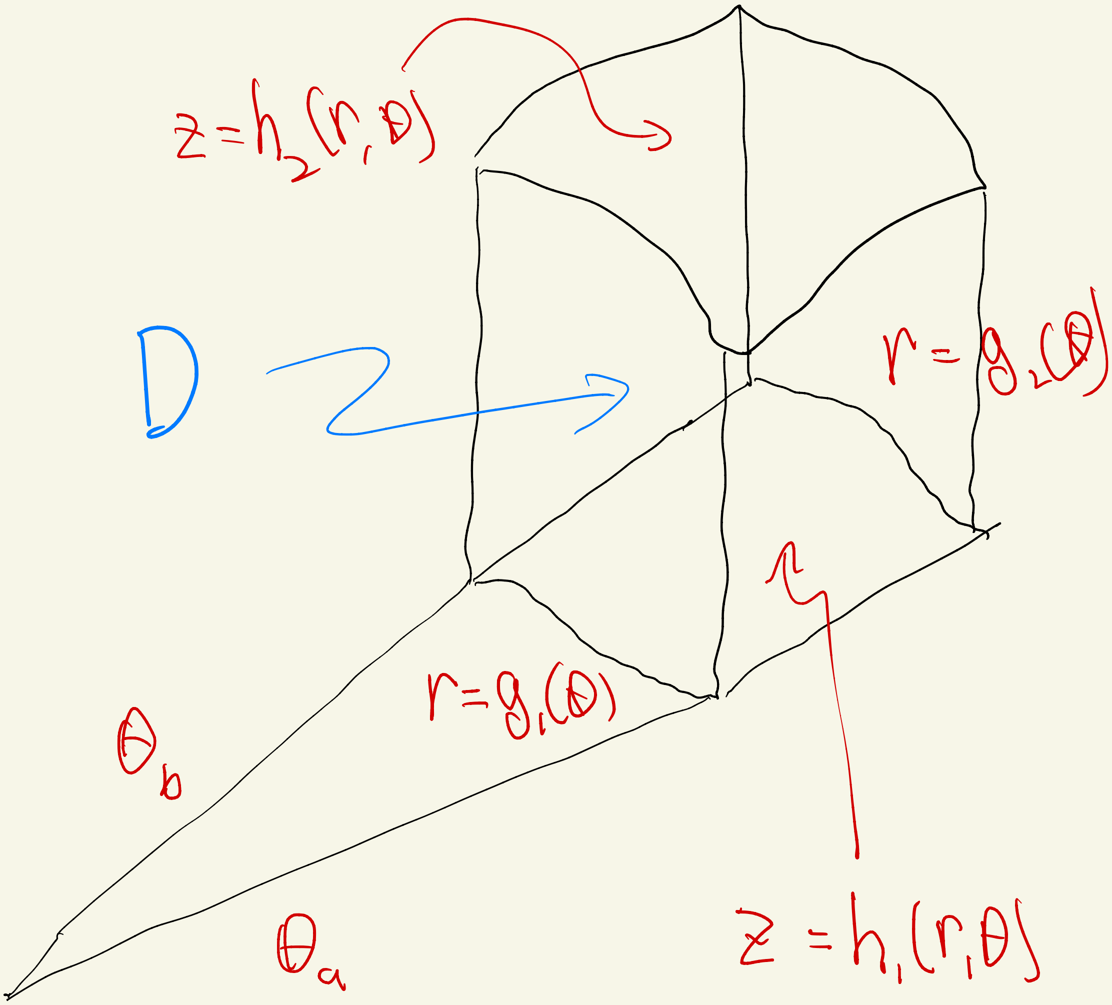
Thm Cylindrical Coordinates

$$\iiint_{D_{xyz}} f(x,y,z)\,dV_{xyz} = \iiint_{D_{r\theta z}} f(r\cos\theta, r\sin\theta, z)\, r \, dV_{r\theta z}$$

Amplification factor for volume

If the boundaries line up nicely enough, you can iterate the integral to get an exact value

Eg



$$\iiint_D f(x,y,z)\,dV = \iiint_{D_{r\theta z}} \bar{f}(r,\theta,z)\,r\,dr\,d\theta\,dz$$

$$= \int_{\theta_a}^{\theta_b} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

☒ Same idea different geometry for Spherical Coordinates

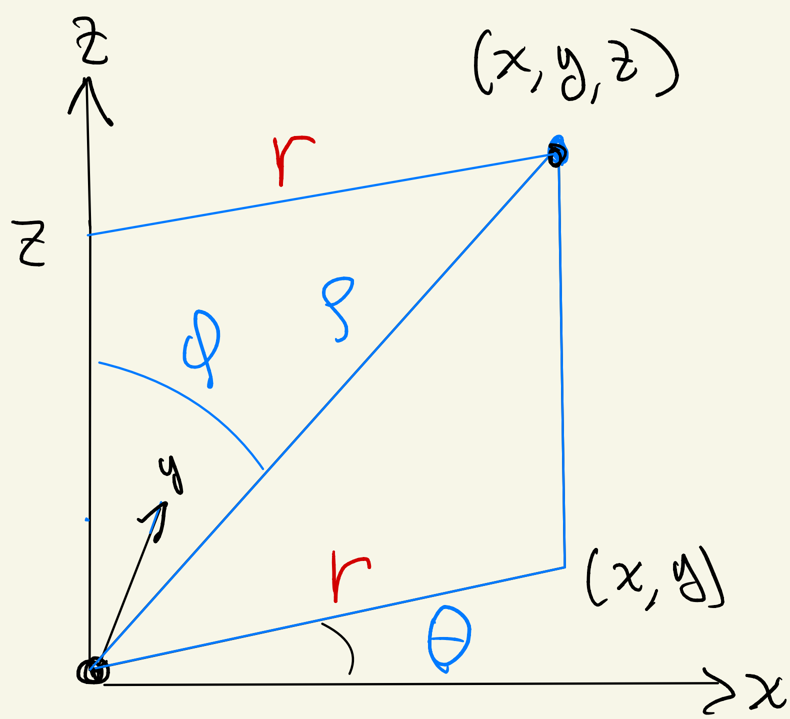
- Spherical Coordinates:

$z = \rho \cos\varphi$

$r = \rho \sin\varphi$

$x = r\cos\theta$

$y = r\sin\theta$



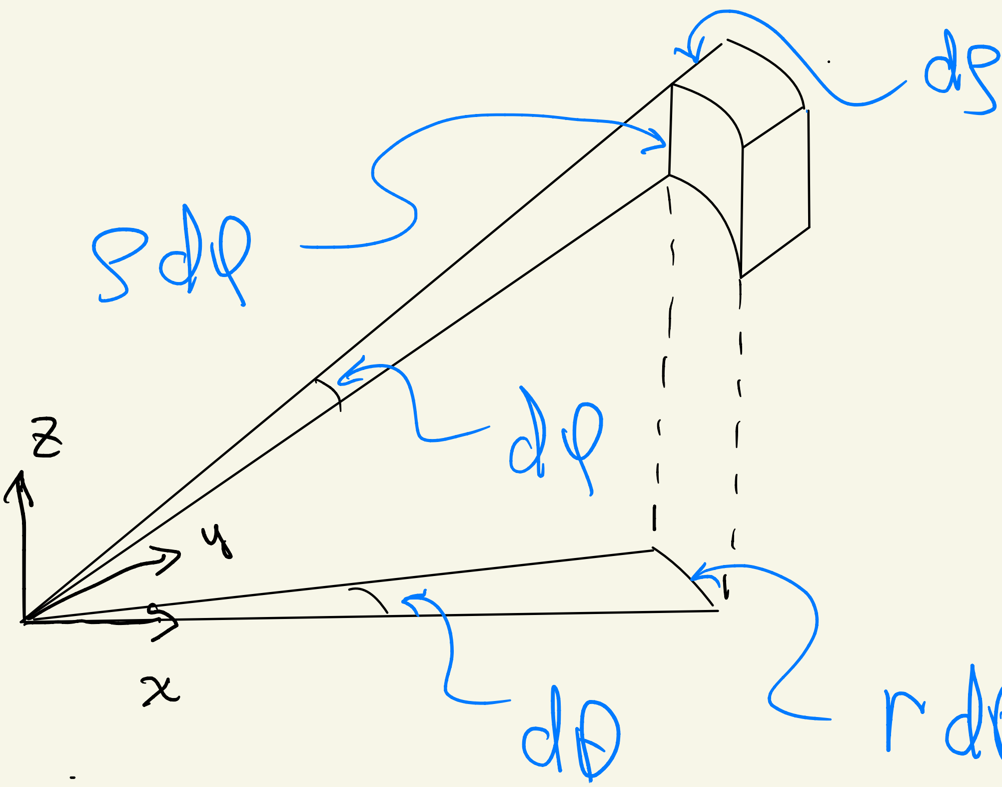
$$\boxed{\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}}$$

Q: What is the amplification factor for volume?

- **Q: What is A**

$$dx\,dy\,dz = \underbrace{A}_{\text{Amplification factor for volume}}\,d\rho\,d\varphi\,d\theta$$

We get A from **geometry**:



$r\,d\theta = \rho \sin\varphi\,d\theta$

$$dx\,dy\,dz = r\rho\,d\rho\,d\varphi\,d\theta = \underbrace{\rho^2 \sin\varphi}_{A}\,d\rho\,d\varphi\,d\theta$$

Theorem: Spherical Coordinates

$$\iiint_{D_{xyz}} f(x,y,z)\, dV$$

$$= \iiint_{D_{\rho\varphi\theta}} \bar{f}(\rho,\varphi,\theta)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$A = \rho^2 \sin\varphi$

$$\bar{f}(\rho,\varphi,\theta) = f\underbrace{(x,y,z)}_{(\rho\cos\theta\sin\varphi,\ \rho\sin\theta\sin\varphi,\ \rho\cos\varphi)}$$

For simple enough functions with simple enough geometry you can get an exact value by iterating the integral...

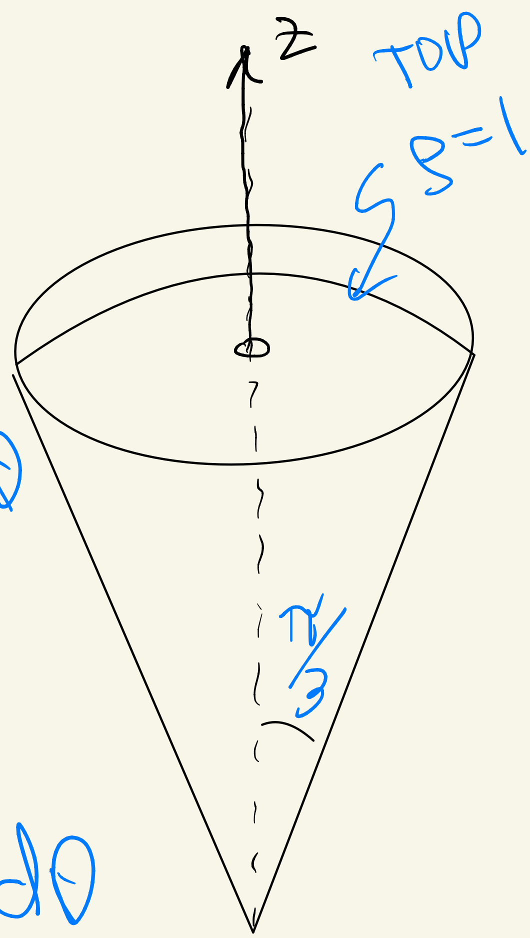
① **Example**: Find the volume of the ice-cream shaped cone D cut from $\rho \le 1$ by cone $\varphi = \pi/3$

Soln: $V = \iiint_D 1 \cdot dV$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \sin\varphi \left.\frac{\rho^3}{3}\right]_0^1 d\varphi \, d\theta$$

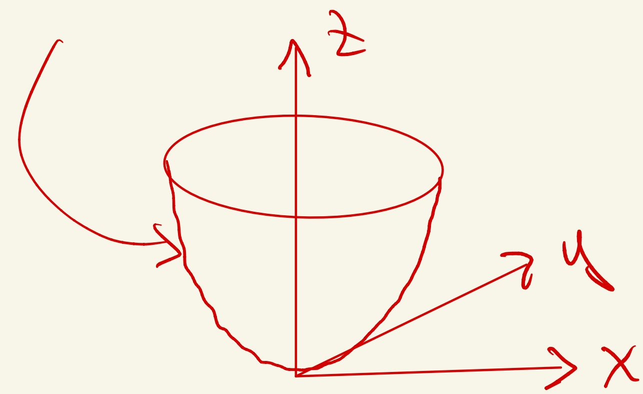
$$= \frac{1}{3}\int_0^{2\pi} \underbrace{-\cos\varphi\Big]_0^{\pi/3}}_{-\cos\frac{\pi}{3} + \cos 0 = -\frac{1}{2}+1 = \frac{1}{2}} d\theta = \frac{1}{3}\cdot\frac{1}{2}\cdot 2\pi = \boxed{\frac{\pi}{3}}$$

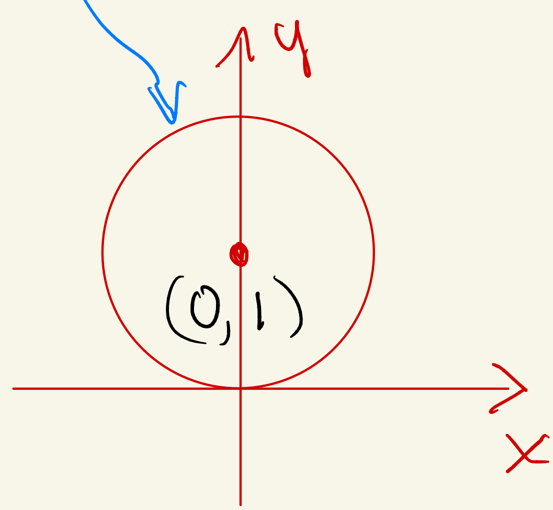


(2) Iterate (but don't evaluate) the integral $I=\iiint\limits_{D} f(r,\theta,z)\,dV$ where D is the region bded below by plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$ and above by paraboloid $z=x^2+y^2$

$x^2+(y-1)^2=1$ circle center $(x,y)=(0,1)$

$z=x^2+y^2$ paraboloid

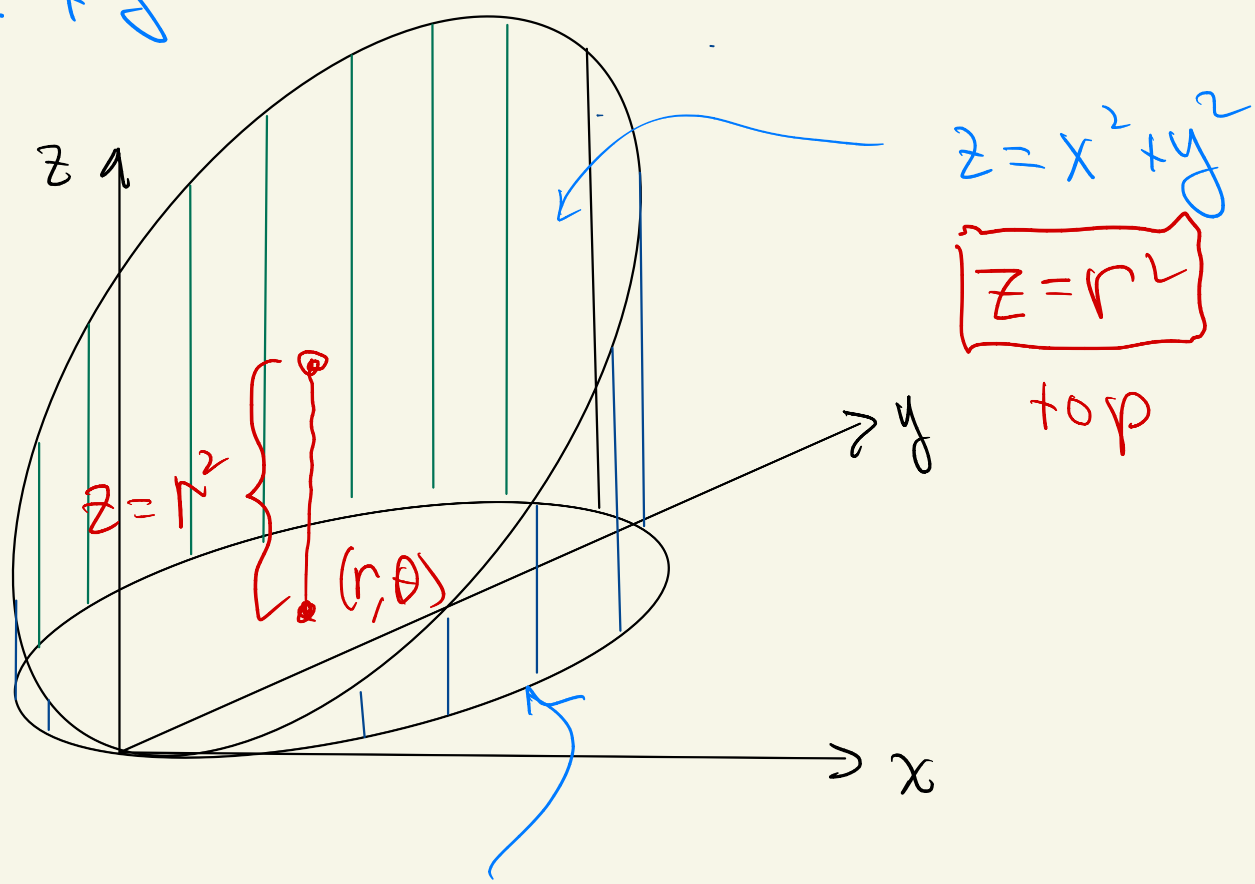




$x^2+(y-1)^2=1$

$z=x^2+y^2$

Graph:



$x^2+y^2-2y+\cancel{1}=\cancel{1}$

$r^{\cancel{2}}=2y=2\cancel{r}\sin\theta$

$r=2\sin\theta \qquad 0\le\theta\le\pi$

$$I=\int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$